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# A METHOD FOR OBTAINING CONSTANTS FOR FORMULAS OF ORGANIC GROWTH*

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The growth of many organisms may be computed from the equation

$$\log \frac{x}{A - x} = K(t - t_1),$$

in which $x$ represents the size of the organism at time $t$, $A$ represents the size of the organism at maturity or at the end of a particular cycle of growth, $t_1$ is the time at which $x = A/2$, and $K$ is a constant. This is the equation for an autocatalytic reaction and we are indebted to Robertson (1908) for showing its applicability to growth processes.

While the equation expresses the growth of plants with a high degree of accuracy, it often fails to fit the observed data in the early life of the organism. In other words, the computed values of $x$ are too large when $t$ is small. Recently this feature of the equation has been discussed by Mitscherlich (1919) and Rippel (1919).

The present paper intends to show how a simple graphic method may be used to overcome the difficulty just stated.

Let us assume that the true state of affairs is represented by the equation

$$\log \frac{x}{A - x} = K(t - t_1)^c.$$

This is merely assuming that the exponent of $(t - t_1)$ may or may not be unity. We may write

$$\log \left(\log \frac{x}{A-x}\right) = \log K + c \log (t - t_1)$$

This is the equation of a straight line if $\log \left(\log \frac{x}{A-x}\right)$ be used as ordinate and $\log (t - t_1)$ as abscissa. The intercept on the $y$-axis will be $\log K$ and the slope of the line $c$.

When $t < t_1$ the values of $(t - t_1)$ are negative in sign, as $t$ approaches the value of $t_1$ the value of $(t - t_1)$ decreases to zero. As $t$ increases beyond $t_1$ the values of $(t - t_1)$ are positive. However the sign of the quantity $(t - t_1)$ does not affect the logarithm. The result is that we actually get two lines on the chart, one for values of the equation when $t < t_1$ and another for values when $t > t_1$. These lines as determined from the observations, are seldom superimposed. With them as guides the "best" straight line is drawn from which the values of the slope and intercept are read off. If the lines make much of an angle with each other it usually means that the value of $t_1$ has been poorly chosen. The choice of a value for $t_1$ is facilitated if one ascertains all the values of $\log\left(\log \frac{x}{A-x}\right)$, and from them determines by inspection the values of $(t-t_1)$ which will bring approximately equivalent values of $\log\left(\log \frac{x}{A-x}\right)$ as nearly as possible the same distance from the ordinate. For this purpose, the highest and lowest values of $\log\left(\log \frac{x}{A-x}\right)$ are less reliable than the intermediate values.

A few examples may be given.

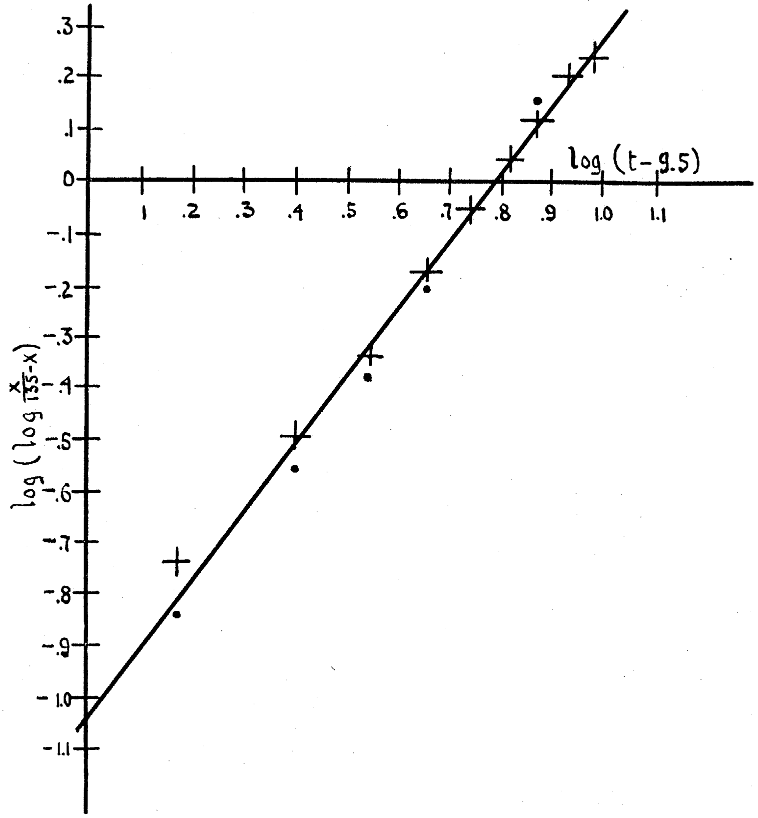

FIG. 1

**Graph of values for lemon shoots. Points represent values of log $(t-9.5)$ when $t < t_1$; crosses represent same when $t > t_1$.**

The average length of 37 shoots on a group of lemon trees is given in the column headed $x$ in table 1. The measurements were made each week for the first 21 weeks of their growth. From the observed length at intervals of a week the values of

$$\log\left(\log \frac{x}{135-x}\right) = \log K + c \log (t - 9.5)$$

were ascertained and plotted (fig. 1). From the line thus obtained it was ascertained that $\log K = -1.05$ and $K = .0891$. The value of $c$ as determined by the slope was 1.35.

Table 1 contains in the column headed $x_1$ the values computed from the equation

$$\log \frac{x}{135 - x} = .0891(t - 9.5)^{1\cdot35}.$$

TABLE 1

GROWTH OF LEMON SHOOTS. COMPARISON OF OBSERVED AND CALCULATED VALUES

| $t$ WEEKS | $x$ CM. | $x_1$ CM. | $\theta_1$ CM. | $x_2$ CM. | $\theta_2$ CM. |
|---|---|---|---|---|---|
| 1 | 3.0 | 3.2 | .2 | 5.3 | 2.3 |
| 2 | 4.9 | 5.7 | .8 | 7.6 | 2.7 |
| 3 | 9.6 | 9.6 | .0 | 10.8 | 1.2 |
| 4 | 15.6 | 12.8 | −2.8 | 15.1 | −0.5 |
| 5 | 25.8 | 23.4 | −2.4 | 20.9 | −4.9 |
| 6 | 37.3 | 33.4 | −3.9 | 28.5 | −8.8 |
| 7 | 46.8 | 44.8 | −2.0 | 37.8 | −9.0 |
| 8 | 56.5 | 55.6 | −0.9 | 48.9 | −7.6 |
| 9 | 65.2 | 61.4 | −3.8 | 61.2 | −4.0 |
| 10 | 73.5 | 73.6 | .1 | 73.8 | 0.3 |
| 11 | 81.6 | 79.4 | −2.2 | 86.1 | 4.5 |
| 12 | 91.4 | 90.2 | −1.2 | 97.2 | 5.8 |
| 13 | 100.0 | 101.7 | −1.7 | 106.5 | 6.5 |
| 14 | 110.9 | 111.6 | .7 | 114.1 | 3.2 |
| 15 | 119.4 | 122.2 | 2.8 | 119.9 | 0.5 |
| 16 | 125.3 | 125.4 | .1 | 124.2 | −1.1 |
| 17 | 128.8 | 129.3 | .5 | 127.4 | −1.4 |
| 18 | 131.6 | 131.8 | .2 | 129.7 | −1.9 |
| 19 | 132.5 | 133.1 | .6 | 131.4 | −1.1 |
| 21 | 134.3 | 134.5 | .2 | 133.2 | −1.1 |
| Root-mean-square deviation | | | 1.88 | | 4.40 |

The root-mean-square deviation of calculated from observed values was 1.88 cm.

In the column headed $x_2$ will be found values computed from the equation

$$\log \frac{x}{135-x} = .1635(t - 9.5).$$

The root-mean-square deviation of these values is 4.40 cm.

TABLE 2

GROWTH OF SHOOTS ON YOUNG PEAR TREES

| $t$ DAYS | $x$ CM. | $x_1$ CM. | $\theta_1$ CM. | $x_2$ CM. | $\theta_2$ CM. |
|---|---|---|---|---|---|
| 0 | | 6.8 | 6.8 | 8.1 | 8.1 |
| 7 | 5 | 10.0 | 5.0 | 10.7 | 5.7 |
| 14 | 11 | 14.6 | 3.6 | 15.4 | 4.4 |
| 21 | 19 | 20.7 | 1.7 | 21.3 | 2.3 |
| 28 | 29 | 28.6 | −0.4 | 28.8 | −0.2 |
| 35 | 40 | 39.2 | −0.8 | 38.1 | −1.9 |
| 42 | 48 | 49.8 | 1.8 | 48.5 | 0.5 |
| 49 | 59 | 59.0 | 0. | 54.4 | −4.6 |
| 55 | 69 | 67.6 | −1.4 | 69.0 | 0. |
| 63 | 76 | 79.1 | 3.1 | 80.4 | 4.4 |
| 70 | 86 | 88.2 | 2.2 | 88.8 | 2.8 |
| 77 | 94 | 95.5 | 1.5 | 95.6 | 1.6 |
| 84 | 100 | 101.5 | 1.5 | 100.9 | 0.9 |
| 91 | 102 | 105.5 | 3.5 | 104.8 | 2.8 |
| 98 | 108 | 108.5 | 0.5 | 107.6 | −0.4 |
| 105 | 110 | 110.5 | 0.5 | 109.6 | −0.4 |
| 112 | 111 | 111.8 | 0.8 | 110.9 | −0.1 |
| 119 | 112 | 112.5 | 0.5 | 111.9 | −0.1 |
| 126 | 113 | 113.0 | 0. | 112.6 | −0.4 |
| Root-mean-square deviation | | | 2.38 | | 3.14 |

In a recent paper (Reed, 1920) I showed that the growth of shoots on Bartlett pear trees followed the course of an autocatalytic reaction very closely, except in the first few weeks of the season. During the early part of the growth period the calculated values were larger than those observed. Using the observed data I have computed values of $K$ and $c$ from the equation

$$\log\left(\log\frac{x}{114 - x}\right) = \log K + c \log (t - 47.4).$$

Plotting the values of this equation (fig. 2) the value of $\log K$ (the intercept) is $-1.75$ ($K = .0178$) and $c$ (the slope) is 1.09.

When the size of the pear shoots is computed from the equation

$$\log \frac{x}{114 - x} = .0178(t - 47.4)^{1.09}.$$

we obtain values agreeing more closely with the observed, as shown by table 2, where $x$ = observed mean length of the young shoots at successive intervals; $x_1$ = length calculated from

$$\log \frac{x}{114 - x} = .0178(t - 47.4)^{1.09}.$$

$\theta_1$ = deviations of $x_1$ from $x$; $x_2$ = length calculated from

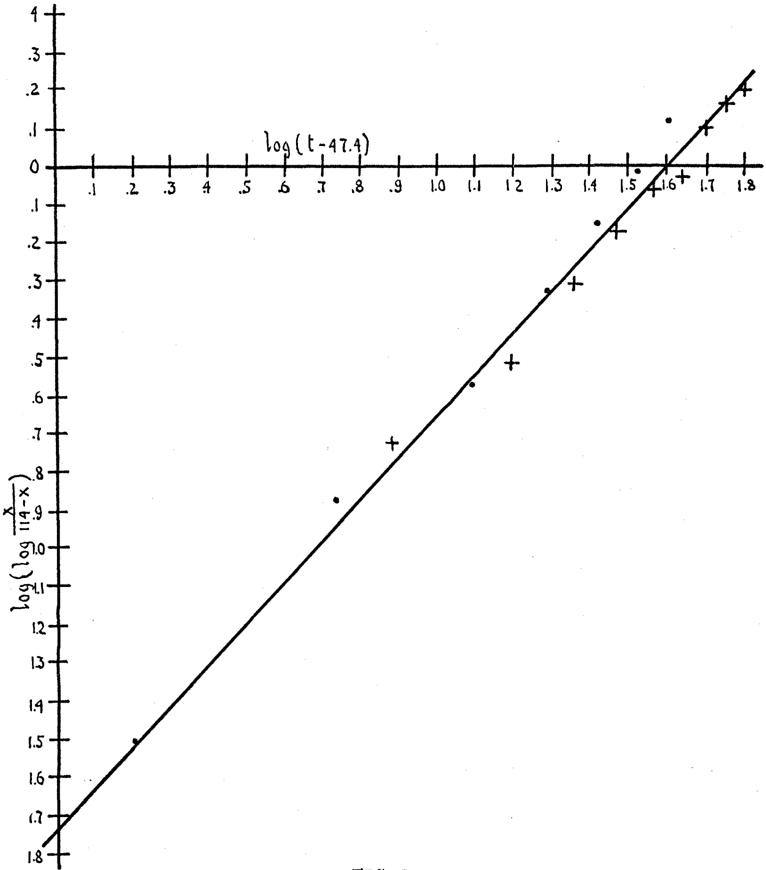

FIG. 2

Graph of values for pear shoots. Points represent values of log $(t-47.4)$ when $t < t_1$; crosses represent same when $t > t_1$.

$$\log\frac{x}{114-x} = .0242(t-47.4)$$

$\theta_2$ = deviations of $x_2$ from $x$. The root-mean-square deviation of $x_1$ from $x$ is 2.38 cm., and that of $x_2$ from $x$ is 3.14.

Finally, a growth process which closely coincided with values computed from the usual formula, may be mentioned. In the paper cited (Reed, 1920) the average height of a population of young walnut trees (*J. regia*) was presented. The calculated values had a root-mean-square deviation from the observed of only 3.12 cm. The equation for growth as derived by graphic method was

$$\log\frac{x}{153-x} = .0208(t-96)^{1.03}.$$

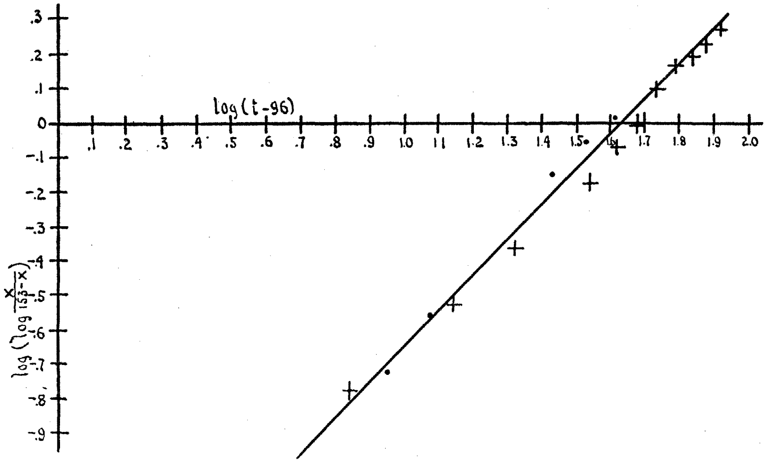

FIG. 3

Graph of values for walnut trees. Points represent values of log $(t - 96)$ when $t < t_1$; crosses represent same when $t > t_1$.

The value of $K$ here is only .002 less than that first used and the exponent of $(t - 96)$ is practically unity. The root-mean-square deviation of values obtained from this equation was 2.65 cm., which differs so slightly from the first that there is no real advantage in favor of one equation over the other.

The use of the method described does, however, give a more accurate means of computing the growth in cases where the initial growth of the organisms falls measurably below that given by the equation as previously used.

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